Definition $f: E \mapsto E'$. f is uniformly continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

Remark: δ is independent of the choice of x.

Example: (1) $f(x) = 3x$

Let $\varepsilon > 0$, $\delta = \frac{\varepsilon}{3}$

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| = 3|x-y| < 3\delta = \varepsilon$$

So, $f(x)$ is uniformly continuous

(2) $f(x) = x^2$ is continuous but not uniformly continuous.

~~proof~~: Assume $x > \delta$ and $|x-y| < \delta$,

$$|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y|$$
$$|x-y| \, (2x - \delta)$$

Let $\varepsilon > 0$, $\delta > 0$. Select $y = x + \frac{\delta}{2}$, then $|f(x)-f(y)| = |x-y||x+y|$

$$= \delta/2\,(2x+\tfrac{\delta}{2})$$

$$> \delta x$$

If $x > \frac{\varepsilon}{\delta}$, $|f(x)-f(y)| > \varepsilon$, even though $|x-y| = \frac{\delta}{2} < \delta$.

<u>Remark</u>: $f(x)$ is uniformly continuous

$\Rightarrow$ $f'(x)$ is bounded.

$|f(x)-f(y)| = f'(t)\,|x-y|$, $t \in [x,y]$.

(3) $f(x) = \sqrt{x}$, $f: [0,+\infty) \mapsto [0,+\infty)$

is uniformly continuous.

~~proof~~: Let $\varepsilon > 0$,

$$|\sqrt{x}-\sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \le \frac{|x-y|}{\sqrt{|x-y|}+\sqrt{|x+y|}}$$

$$= \frac{1}{2}\sqrt{|x-y|} < \varepsilon.$$

if $\delta = 4\varepsilon^2$.

<u>Obs</u>: Uniformly continuous $\Rightarrow$ continuous

<u>Theorem</u>: $f: E \mapsto E'$, E is compact.

f continuous $\Longleftrightarrow$ f uniformly continuous.

<u>proof</u>: Let $\varepsilon > 0$, $\forall x \in E$, $\exists \delta_x > 0$

s.t. $f(B_{\delta_x}(x)) \subset B_{\frac{\varepsilon}{2}}(f(x))$

so, $E = \bigcup\limits_{x \in E} B_{\frac{\delta_x}{2}}(x)$.

Since E is compact. $\exists x_1, \cdots, x_n$, s.t.

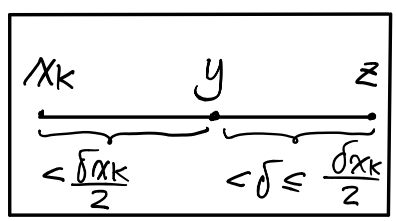
$$E = \bigcup_{i=1}^{n} B_{\frac{\delta_{x_i}}{2}}(x_i)$$

Select $\delta = \min\limits_i \{ \frac{\delta_{x_1}}{2}, \cdots, \frac{\delta_{x_n}}{2} \}$.

Let $y, z \in E$, s.t. $d(y, z) < \delta$.

So, $y \in B_{\frac{\delta_{x_k}}{2}}(x_k)$

since $d(y, z) < \delta \le \frac{\delta_{x_k}}{2}$



$$d(x_k, z) \le d(x_k, y) + d(y, z)$$
$$< \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}$$

i.e. $z \in B_{\delta_{x_k}}(x_k)$. $y \in B_{\frac{\delta_{x_k}}{2}}(x_k) \subset B_{\delta_{x_k}}(x_k)$.

Since f is continuous. f is continuous at x_k.

$$\Rightarrow f(B_{\delta_{x_k}}(x_k)) \subset B_\varepsilon(f(x_k))$$

So, $d(f(y), f(z)) \leq d(f(y), f(x_k))$
$$+ d(f(x_k), f(z))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Example: $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$

is uniformly continuous.

proof: $\begin{cases} f(x) \text{ is continuous} \\ [0,1] \text{ is closed and bounded} \end{cases}$

$\Rightarrow [0,1]$ is compact.

Theorem $f: E \mapsto E'$ continuous.

E is connected $\Rightarrow f(E)$ is connected.

proof: Assume $f(E) \subset U_1' \amalg U_2'$, U_1, U_2 open.

then $E \subset f^{-1}(U_1') \amalg f^{-1}(U_1')$

and both $f^{-1}(U_1')$ and $f^{-1}(U_2')$ are open.
so, $f^{-1}(U_1')$ is also closed (its compliment $f^{-1}(U_2')$ is open).

Since E is connected, $f^{-1}(U_1')$ is either E or ϕ.
If $f^{-1}(U_1') = E$, then $f(E) \subset U_1'$
If $f^{-1}(U_1') = E$, then $f(E) \subset U_2'$.
Thus, $f(E)$ is also connected.

<u>Corollary</u>: $f : [a, b] \mapsto \mathbb{R}$, f continuous.
Assume $f(a) < f(b)$, Let $\eta \in (f(a), f(b))$
then $\exists\, c \in (a, b)$ s.t. $f(c) = \eta$.

~~proof~~: $U_1 = \{ y \in \mathbb{R} : y > \eta \}$ open
$U_2 = \{ y \in \mathbb{R} : y < \eta \}$. open.
$\Rightarrow f(a) \in U_1$ and $f(b) \in U_2$.

and $U_1 \cap U_2 = \emptyset$.

$$\begin{cases} f([a,b]) \cap U_1 \neq \emptyset \\ f([a,b]) \cap U_2 \neq \emptyset. \end{cases}$$

but $[a,b]$ connected $\Rightarrow$ $f([a,b])$ connected.

$f([a,b]) \not\subset U_1 \amalg U_2$

Then, $\exists\, c$ s.t. $f(c) \notin U_1 \amalg U_2$

Thus, $f(c) = \eta$.

<u>Lemma</u>: $x, y \in \mathbb{R}^n$, $f: [0,1] \mapsto \mathbb{R}^n$

$f(t) = x + t(y - x)$ line segment between x and y.

$\Longrightarrow$ f is continuous.

~~proof~~: Let $t, s \in [0,1]$.

$d(f(t) - f(s)) = |t-s| \cdot \underbrace{\sqrt{(y_1-x_1)^2 + \cdots + (y_n-x_n)^2}}_{< k}$

select $\delta = \frac{\varepsilon}{k}$.

Lemma: $S_{xy} = f([0,1])$ where $f(t) = x + t(y-x)$. S_{xy} is connected.

~~proof~~: $\left.\begin{array}{l} f \text{ is continuous} \\ [0,1] \text{ is connected} \end{array}\right\} \Rightarrow f([0,1])$ is connected.

Proposition In $\mathbb{R}^n$, $B_r(x)$ is connected. $\forall r>0$ and $x \in \mathbb{R}^n$.

~~proof~~: (1) $y \in B_r(x) \Rightarrow S_{xy} \subset B_r(x)$.

(2) $B_r(x) = \bigcup_{y \in B_r(x)} S_{xy}$.

$\left\{\begin{array}{l} \text{Each } S_{xy} \text{ is connected,} \\ x \in S_{xy}, \ \forall y \in B_r(x) \end{array}\right.$

since the union of connected sets that share a common point is connected.

$\Rightarrow B_r(x)$ is connected.